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Some More Universal Scaling Laws for Critical Mappings

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We study such nonlinear mappings $x_{n+1} = F(x_n; b_{cr})$ of an interval I into itself for which the Feigenbaum scaling laws hold (i.e., for which b_{cr} is an accumulation point of bifurcation points). Let x_0 be a random variable with some absolutely continuous distribution in I. We show in particular that (i) the geometric average distance of x_n from the nearest point of the attractor decreases like $n^{-1.93387}$; (ii) the geometric average of $|\partial x_n/\partial x_0|$ increases like $n^{0.60}$; (iii) the geometric mean distance $|x_n - y_n|$ between the iterates of two close-by points x_0, y_0 asymptotically tends towards a value $\sim|x_0 - y_0|^{0.77}$. These—and other—properties are also borne out from a simple probabilistic model which depicts the evolution as a random walklike process.

KEY WORDS: One-dimensional maps; onset of turbulence; Feigenbaum scaling laws; critical phenomena; universality; sensitivity to initial conditions; approach towards fractal attractors.

1. INTRODUCTION

Simple nonlinear mappings

$$x_{n+1} = F(x_n; b) \tag{1.1}$$

of an interval I into itself have a number of very remarkable properties which have recently been the subject of numerous studies.[(1-9)] Two typical examples, studied in this paper, are

$$x_{n+1} = bx_n(1 - x_n) \tag{1.2}$$

and

$$x_{n+1} = bx_n(1 - x_n^2) \tag{1.3}$$

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Both examples have infinitely many sequences of bifurcation points $\{b_{i,k}; k = 0, 1, 2, 3, \ldots\}$ at which a stable cycle of period $i \times 2^k$ turns into one of period $i \times 2^{k+1}$. At the limit points

$$b_{i,\text{cr}} = \lim_{k\to\infty} b_{i,k} \tag{1.4}$$

the attractor is no longer periodic but is a Cantor set, and one finds scaling laws[5,6] which are very reminiscent of critical phenomena.

These laws are universal at least within the class of functions $F(x; b)$ which are strictly monotonic in x except for a single parabolic maximum. But they also seem to apply at the onset of turbulence in many physical phenomena such as, e.g., the Benard problem.[10]

A typical feature of critical phenomena is that laws which are exponential away from the critical point become power laws, with universal anomalous exponents, at the critical point. If the above analogy is not superficial, we should observe similar power laws for critical mappings as well.

In the present paper, we shall study three closely related aspects of critical mappings. These are the sensitivity to initial conditions, the speed of approach towards the attractor, and the behavior of the derivative $\partial x_n(x_0, b = b_{\text{cr}})/\partial x_0$, all of them for $n \to \infty$.

In the *periodic regions*, they are trivially related. Let us define as usual the Lyapounov characteristic exponent as

$$\chi(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln\left|\frac{\partial x_n}{\partial x_0}\right| \tag{1.5}$$

Since the maps considered are ergodic, with the attractor consisting of a finite number of points, $\chi(x)$ is, for nearly all x, independent of x and negative. The distance between the iterates of (nearly) any two sufficiently close-by points x_0 and y_0 thus decreases like

$$\Delta_n \equiv |x_n - y_n| \underset{n\to\infty}{\sim} e^{n\chi}, \qquad \chi < 0 \tag{1.6}$$

Taking for y_0 a point of the attractor, one finds that also

$$d_n = \inf_{y \in \text{attractor}} |x_n - y| \sim e^{n\chi} \tag{1.7}$$

In the *chaotic regions*, the situation is similar. Again we have

$$\left|\frac{\partial x_n}{\partial x_0}\right| \sim e^{n\chi} \tag{1.8}$$

for nearly all x_0, this time with $\chi > 0$. The distance between two arbitrary close points starts to increase according to Eq. (1.6). However, when Δ_n becomes of the same order of magnitude as some characteristic length of the attractor, this increase has to stop and the average of Δ_n stays independent of n during the further evolution. Remember that the attractor in the chaotic region consists in general of a set of disjoint intervals, the lengths of which determine this characteristic length. At the critical points, there is no analogous length scale, and $\lim_{n\to\infty}\langle\Delta_n\rangle$ must be determined by Δ_0 (and by the total size of the attractor basin).

Finally, consider the average distance from the attractor: for large n, the chance that x_n does not fall onto the attractor decreases exponentially. Thus, the arithmetic mean distance decreases exponentially, too. The geometric mean distance, however, is exactly zero for sufficiently large n.

As we have already said, we expect that the above exponential laws turn into power laws *at the critical point*. Thus we propose

$$\left|\frac{\partial x_n}{\partial x_0}\right| \sim n^{\gamma} \tag{1.9}$$

$$\Delta_n \sim \begin{cases} \Delta_0^{\delta} & \text{for } n\to\infty \\ \Delta_0\cdot n^{\gamma} & \text{for } 1 \ll n \ll \Delta_0^{(\delta-1)/\gamma} \end{cases} \tag{1.10}$$

and

$$d_n \sim n^{-\epsilon} \tag{1.11}$$

with three different universal exponents γ, δ, and ϵ.

However, as we have seen, we must not, in general, expect the same behavior for geometric and for arithmetic mean values, indicating strong and irregular dependence on x_0 and/or on n. Indeed, a few short runs with any pocket calculator show that one has to be very careful in formulating relations like (1.9)–(1.11) more precisely. A typical plot of $|\partial x_n/\partial x_0|$ for Eq. (1.2) with $b = b_{1,\mathrm{cr}} = 3.5699456\ldots$ for fixed x_0 and $1 \leqslant n < 1000$, is shown in Fig. 1. One sees a very erratic behavior. The (geometric) Cesaro-type average

$$\left|\prod_{m=1}^{n} \frac{\partial x_m}{\partial x_0}\right|^{1/n} \tag{1.12}$$

shows, of course, a much smoother deportment, but still a power behavior is far from evident in Fig. 1.

That Eq. (1.9) must be taken with care is also seen from the estimate for the arithmetic average of $|\partial x_n/\partial x_0|$,

$$\int dx \left|\frac{\partial x_n}{\partial x_0}\right| \underset{n\to\infty}{\sim} e^{\mathrm{const}(\ln n)^2} \tag{1.13}$$

which is proven in the Appendix.

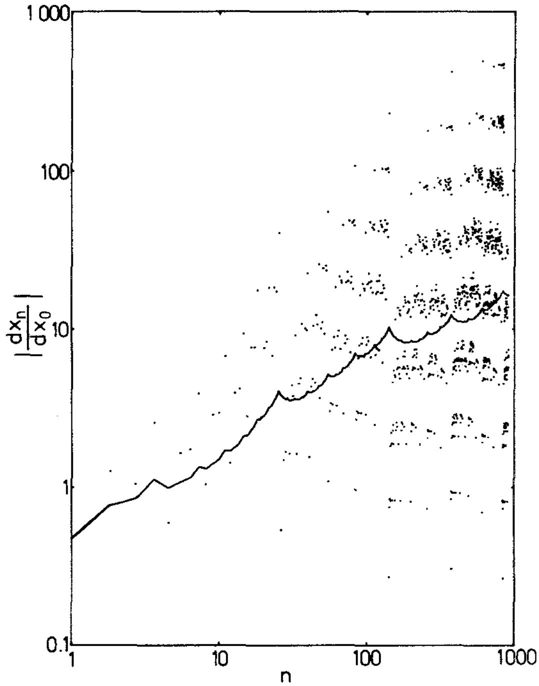

Fig. 1. Values of $|\partial x_n/\partial x_0|$ for a randomly chosen x_0 (= 0.1398025 . . .) for Eq. (1.2) with $b = b_{1,\text{cr}} = 3.56994567$ . . . . The continuous curve represents the (geometric) Cesaro average, Eq. (1.12).

We shall argue in the following that Eqs. (1.9)–(1.11) are valid and universal if they are understood as estimates for the geometric averages, with x_0 distributed according to any absolutely continuous distribution. The critical exponents are

$$\begin{aligned} \epsilon &= 1.9338710 \\ \gamma &= 0.60 \pm 0.01 \\ \delta &= \frac{\epsilon}{\epsilon + \gamma} = 0.77 \pm 0.01 \end{aligned} \tag{1.14}$$

The way in which x_n approaches the (critical) attractor will be discussed in the next section. In Section 3, the behavior of $|\partial x_n/\partial x_0|$ and the sensitivity to initial conditions will be investigated. In both of these sections, only heuristic arguments and Monte Carlo calculations are used to support our claims. Thus any additional piece of evidence in favor of them would be very welcome.

In Section 4, a random-walk-type model will be discussed which displays all essential features discussed above. Notice that critical mappings have no sensitive dependence on initial conditions in the sense of Guckenheimer.[11] Nevertheless, on a more detailed level (as seen, e.g., in the behavior of $\partial x_n/\partial x_0$) they are sensitive, and correspondingly a probabilistic model is very natural.

2. THE APPROACH TOWARDS A CRITICAL ATTRACTOR

The attractor at a critical point is well known to be a Cantor set.[2] Let ξ_0 be the point at which $F(x)$ has its maximum. We shall assume $\xi_0 = 0$, which can always be achieved by a suitable shift of the origin. Then the attractor is just the set[9] $\{\xi_n \mid n = 1, 2, 3, \ldots\}$, where ξ_n is defined recursively through the equation

$$\xi_n = F(\xi_{n-1}), \qquad n \geqslant 1 \tag{2.1}$$

As an example, the attractor of the logistic equation (1.2) at $b = b_{1,\mathrm{cr}}$ is shown in Fig. 2.

Without loss of generality, we shall discuss only limit points of period 2^k cycles. Cycles of period $i \times 2^k$ can be considered as period 2^k cycles of the ith iteration $F^{(i)}(x)$ of $F(x)$. An attractor at a $b_{i,\mathrm{cr}}$ with $i \neq 1$ consists thus of i components for each of which the following discussion applies.

Associated with this attractor is, as with any Cantor set, a hierarchy of intervals. Let us define the open intervals ($k = 0, 1, 2, \ldots$; $i = 1, 2, 3, \ldots 2^k$)

$$I_{k,i} = \begin{cases}]\xi_{2\cdot 2^k+i}, \xi_{3\cdot 2^k+i}[\\ ]\xi_{3\cdot 2^k+i}, \xi_{2\cdot 2^k+i}[\end{cases} \quad \text{if } \xi_{2\cdot 2^k+i} \lessgtr \xi_{3\cdot 2^k+i} \tag{2.2}$$

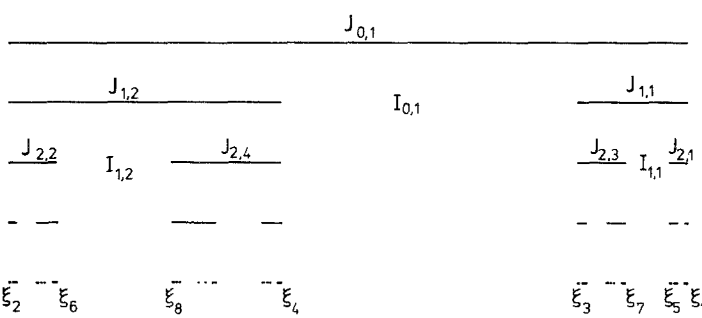

Fig. 2. The attractor of Eq. (1.2) at $b = b_{1,\mathrm{cr}} = 3.56994567\ldots$.

and the closed intervals

$$J_{k,i} = \begin{cases} [\xi_i, \xi_{2^k+i}] \\ [\xi_{2^k+i}, \xi_i] \end{cases} \quad \text{if } \xi_i \lessgtr \xi_{2^k+i} \tag{2.3}$$

More instructive than these formal definitions is, presumably, a look at Fig. 2. Furthermore, we define

$$I_k = \bigcup_{i=1}^{2^k} I_{k,i} \tag{2.4}$$

$$J_k = \bigcup_{i=1}^{2^k} J_{k,i} \tag{2.5}$$

From Fig. 2 it is obvious that

$$J_l = J_k \cup \bigcup_{k'=l}^{k-1} I_{k'} \qquad \text{for } k > l \tag{2.6}$$

The attractor can be characterized as

$$J_0 - \bigcup_{k=0}^{\infty} I_k = \bigcap_k J_k =: J_\infty \tag{2.7}$$

Under the action of F, the intervals $J_{k,i}$ with $i < 2^k$ are mapped into $J_{k,i+1}$, while $J_{k,2^k} \to J_{k,1}$. Thus, if $x \in J_k$, $F(x)$ will also be in J_k. On the other hand, it may happen that $F(x) \in J_k$ although $x \notin J_k$. Thus every domain J_k acts like an absorbing state in the theory of Markov processes, attracting in this way nearly all trajectories.

The only trajectories not attracted into any given J_k are the (repulsive) periodic orbits. Every domain I_k contains exactly one such orbit of period 2^k, with exactly one point $x^*_{k,i}$ in each $I_{k,i}$.

Assume now that x_0 is distributed in J_0 according to some absolutely continuous distribution $w_0(x_0)$. We are interested in the distribution w_n of x_n for very large n, for which we have the recursion formula

$$w_{n+1}(x_{n+1}) = \int dx_n\, w_n(x_n)\delta(x_{n+1} - F(x_n)) \tag{2.8}$$

As was shown by Feigenbaum,[6] the universality properties of critical mappings follow from the "renormalization group" equation

$$F^{(2^{k+1})}(x) \approx -\frac{1}{\alpha} F^{(2^k)}(\alpha x), \qquad \alpha = 2.502907\ldots \tag{2.9}$$

for $x \approx 0$ and k large. Here, $F^{(n)}$ is the nth iterate of F. It follows that

$$\xi_{2n} \approx -\frac{1}{\alpha}\xi_n \tag{2.10}$$

$$I_{k+1,2i} \approx -\frac{1}{\alpha} I_{k,i} \tag{2.11}$$

and

$$J_{k+1,2i} \approx -\frac{1}{\alpha} J_{k,i} \tag{2.12}$$

for large k, and for n and i containing many factors of 2.

In the following, we shall restrict ourselves to the Feigenbaum map $g(x)$ for which these asymptotic relations hold exactly, i.e., for which in particular

$$g(g(x)) = -\frac{1}{\alpha} g(\alpha x) \tag{2.13}$$

and which is normalized by

$$g(0) = 1 \tag{2.14}$$

Questions of universality will be discussed later.

For the map g, we get

$$w_{n+2}\left(-\frac{z}{\alpha}\right) = \int dx\, w_n\left(-\frac{x}{\alpha}\right)\delta(z - g(x)) \tag{2.15}$$

On the whole real axis this is easily seen to be compatible with

$$w_{2n}(x) = c w_n(-\alpha x) \tag{2.16}$$

for any arbitrary constant c. However, the relation (2.16) implies that $w_n(x)$ is not concentrated in J_0, contrary to our assumption.

Nevertheless, we claim that Eq. (2.16) is asymptotically (for $n \to \infty$) consistent, *provided that* $x \in J_{1,2}$. Take $z \in J_{0,1}$ and assume Eq. (2.16) to be correct for some n. Then $(-z/\alpha) \in J_{1,2}$ and

$$\begin{aligned} w_{2n+2}\left(-\frac{z}{\alpha}\right) &= \int dx\, w_{2n}\left(-\frac{x}{\alpha}\right)\delta[z - g(x)] \\ &= c\int_{x \in J_{0,1}} dx\, w_n(x)\delta(z - g(x)) \\ &\quad + c\int_{-x/\alpha \in I_{0,1}} dx\, w_{2n}\left(-\frac{x}{\alpha}\right)\delta(z - g(x)) \end{aligned} \tag{2.17}$$

In the second integral, we cannot use Eq. (2.16), but asymptotically we expect anyhow that

$$w_{2n}(x) \underset{n\to\infty}{\to} 0 \qquad \text{for } x \in I_{0,1} \tag{2.18}$$

and thus we get $w_{2n+2}(-z/\alpha) \approx c w_{n+1}(z)$, Q.E.D.

Finally, the constant c is fixed by the normalization condition

$$\int dx\, w_n(x) = 1 \tag{2.19}$$

For simplicity, let us assume

$$\int_{x \in J_{1,1}} dx\, w_n(x) = \int_{x \in J_{1,2}} dx\, w_n(x) \tag{2.20}$$

Then

$$\int dx\, w_{2n}(x) = 2\int_{x \in J_{1,2}} dx\, w_{2n}(x)$$
$$= \frac{2c}{\alpha} \int dx\, w_n(x) \tag{2.21}$$

and

$$c = \alpha/2 \tag{2.22}$$

If the condition (2.20) does not hold, we repeat the same argument with $g(x)$ replaced by $g^{(2k)}(x)$, and with $J_{0,1}$ replaced by $J_{k,2^k}$. For sufficiently large k, the analog of Eq. (2.20) will be true to any wanted accuracy.

We now propose that the above is not only compatible with Eq. (2.15), but that indeed it describes the asymptotic behavior for all absolutely continuous initial distributions $w_0(x)$. For an arbitrary critical map, we propose finally that $w_n(x)$ becomes universal for $n \to \infty$ and $x \approx 0$, with

$$w_{2n}(x) \underset{n\to\infty}{\approx} \frac{\alpha}{2} w_n(-\alpha x) \qquad \text{for } x \approx 0 \tag{2.23}$$

We shall check this by the Monte Carlo calculations presented in this and in the following section.

Consider now the integrals

$$P_k(n) = \int_{x \in I_k} dx\, w_n(x) \tag{2.24}$$

They are the probabilities that a randomly chosen point x_0 gives an $x_n \in I_k$. For the Feigenbaum map, the above yields

$$P_k(2n) \approx P_{k-1}(n) \qquad \text{for } n \gg 1 \tag{2.25}$$

This means that essentially the evolution corresponds to a pure shift of the distribution $P_k(n)$ towards larger values of k, with

$$\langle k \rangle \approx \frac{\ln n}{\ln 2} + \text{const} \tag{2.26}$$

For other critical maps, we observe that, for large n, only points $x_n \approx 0$ (where w_n is universal) can jump from some I_k to an $I_{k'}$ ($k' \neq k$) during an iteration. Thus we expect Eqs. (2.25) and (2.26) to be universal, too.

In order to check this numerically, we have performed Monte Carlo calculations with Eq. (2) for $b = b_{1,\text{cr}}$. We have chosen 10^5 starting points $x_1 \in J_{0,1}$ with a flat distribution. The resulting distributions $P_k(n)$, shown in Fig. 3, clearly agree with Eq. (2.25) within the statistical errors, for $n \gtrsim 64$. In this and in similar runs, we have also tried to check Eq. (2.23) directly. The results were fully compatible with it, but the large statistical errors did not allow for quantitative conclusions.

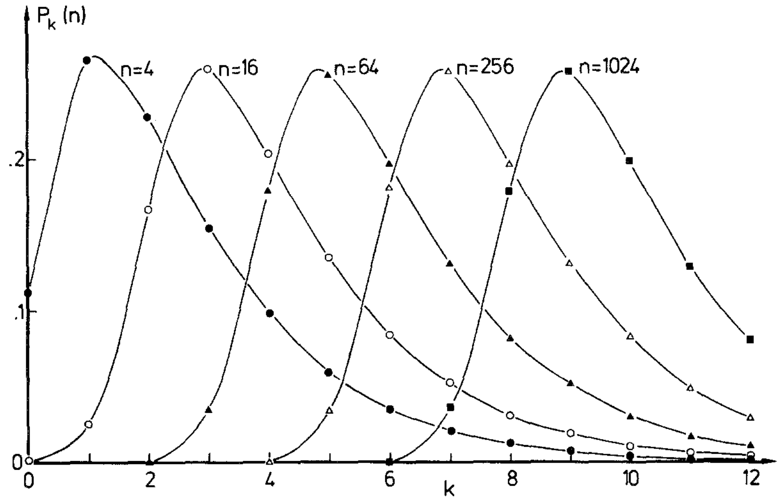

Fig. 3. Probabilities $P_k(n)$ describing the chance that a randomly chosen x_0 will lead to $x_n \in I_k$. The lines connecting the points are drawn only to guide one's eyes.

The decrease of the mean distance from the attractor, Eq. (1.10), now follows straightforwardly. The geometrical average of the lengths of the intervals $I_{k,i}$ for fixed k,

$$\langle l \rangle_k = \prod_{i=1}^{2^k} |I_{k,i}|^{2^{-k}} \tag{2.27}$$

is numerically found to decrease with k like

$$\langle l \rangle_k \underset{k\to\infty}{\sim} \lambda^k \tag{2.28}$$

with $\lambda = 0.26172597 \ldots$. This is clearly seen from the ratios

$$r_k = \langle l \rangle_k / \langle l \rangle_{k-1} \tag{2.29}$$

shown in Table I for Eq. (1.2) with $b = b_{1,\text{cr}}$ and $b = b_{3,\text{cr}} = 3.84943368 \ldots$, and for Eq. (1.3) with $b = b_{1,\text{cr}} = 2.3002283 \ldots$. For $b = b_{3,\text{cr}}$, the attractor consists of three pieces, which for Eq. (1.2) are shown in Fig. 4. Each of these pieces is similar to Fig. 2, and in each one r_k tends towards the universal asymptotic value. From Eq. (2.23), the average distance of x_n from the attractor is proportional to $\langle l \rangle_k$ with $k \sim \ln n / \ln 2$. Thus we obtain

$$d_n \sim n^{-\epsilon} \tag{2.30}$$

with

$$\epsilon = -\frac{\ln \lambda}{\ln 2} = 1.9338710 \ldots \tag{2.31}$$

Table I. $r_k = \langle l \rangle_k / \langle l \rangle_{k-1}$

			Eq. (1.2); $b = b_{3,cr}$		
k	Eq. (1.2); $b = b_{1,cr}$	Eq. (1.3); $b = b_{1,cr}$	Band I	Band II	Band III
2	0.243	0.268	0.248	0.247	0.248
3	0.2649	0.2612	0.2640	0.2641	0.2640
4	0.261297	0.26179	0.26143	0.26141	0.26143
5	0.261787	0.261720	0.261769	0.261771	0.261769
6	0.2617178	0.2617267	0.2617203	0.2617201	0.2617203
7	0.26172707	0.26172590	0.26172675	0.26172676	0.26172674
8	0.261725828	0.261725980	0.261725871	0.261725870	0.261725872
9	0.2617259907	0.2617259711	0.261725986	0.261725986	0.261725986
10	0.26172596942	0.26172597192	—	—	—
11	0.26172597215	0.26172597183	—	—	—
12	0.26172597180	0.26172597184	—	—	—

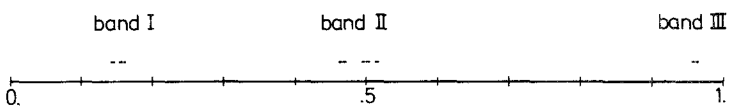

Fig. 4. Attractor of Eq. (1.2) at $b = b_{3,cr}$.

3. SENSITIVITY TO INITIAL CONDITIONS

(a) In this section we shall first discuss the average behavior of $|\partial x_n / \partial x_0|$. As we have already pointed out, we expect a universal power behavior for the geometric average:

$$\left\langle \left| \frac{\partial x_n}{\partial x_0} \right| \right\rangle := \exp\left[\int dx\, w_0(x) \ln \left| \frac{\partial F^{(n)}(x)}{\partial x} \right| \right]$$

$$= \exp\left[\sum_{i=0}^{n-1} \int dx\, w_0(x) \ln \left| F'\left(F^{(i)}(x)\right) \right| \right]$$

$$= \exp\left[\sum_{i=0}^{n-1} \int dx\, w_i(x) \ln |F'(x)| \right] \tag{3.1}$$

with

$$F'(x) = \frac{\partial}{\partial x} F(x; b_{cr}) \tag{3.2}$$

Let us restrict ourselves again to the Feigenbaum map $F = g$, and define

$$a_n = \int dx\, w_n(x) \ln|g'(x)| \tag{3.3}$$

such that

$$\left\langle \left| \frac{\partial x_n}{\partial x_0} \right| \right\rangle = \exp\left(\sum_{i=0}^{n-1} a_i \right) \tag{3.4}$$

Now, using Eqs. (2.8), (2.18), and (2.23), we find

$$\begin{aligned} a_{2n} \approx \frac{\alpha}{2} \int_{x \in J_{1,2}} & dx\, w_n(-\alpha x) \ln|g'(x)| \\ & + \int_{x \in J_{1,2}} dx\, w_{2n-1}(x) \ln|g'(g(x))| \end{aligned} \tag{3.5}$$

For large n, we can assume $w_{2n-1}(x) \approx w_{2n}(x)$ for the same reasons for which we could assume Eq. (2.20), and, using Eq. (2.13), we finally get

$$\begin{aligned} a_{2n} &\approx \frac{\alpha}{2} \int_{x \in J_{1,2}} dx\, w_n(-\alpha x) \ln\left| \frac{d}{dx} g^{(2)}(x) \right| \\ &= \frac{1}{2} a_n \end{aligned} \tag{3.6}$$

This suggests that, on the average,

$$a_n \underset{n \to \infty}{\sim} \frac{\gamma}{n} \tag{3.7}$$

with some constant γ, and we get the proposed scaling law

$$\left\langle \left| \frac{\partial x_n}{\partial x_0} \right| \right\rangle \sim n^{\gamma} \tag{3.8}$$

Unfortunately, we were not able to calculate the critical exponent γ analytically. A very rough estimate is obtained in the following way: we know that $w_n(x)$ is concentrated in domains I_k with $k - (\ln n)/(\ln 2)$ distributed according to a distribution

$$P_k(n) = p\left(k - \frac{\ln n}{\ln 2}\right) \tag{3.9}$$

which can be read off from Fig. 3. We now approximate $w_n(x)$ by a sum of δ functions located at the periodic orbits in I_k, with weights $P_k(n)$. From the renormalization group equation (2.13), one finds

$$\prod_{i=1}^{2^k} |g'(x_{k,i}^*)| = \left| \frac{d}{dx} g^{(2^k)}(x_{k,i}^*) \right| = |g'(x_{0,1}^*)| \tag{3.10}$$

and from Ref. 6 we know that

$$c \equiv |g'(x^*_{0,1})| = 1.60119\ldots \tag{3.11}$$

We then get

$$\gamma \approx \ln c \sum_k \frac{p(k)}{2^k} \approx 0.86 \tag{3.12}$$

In order to obtain a more precise value, and to check the scaling behavior, we again have performed Monte Carlo calculations.

After choosing x_0 at random, we have calculated $\ln|\partial x_n/\partial x_0|$ for n up to $2^{13} = 8192$, and have made linear fits

$$\ln\left|\frac{\partial x_n}{\partial x_0}\right| = \text{const}_N + \gamma_N \ln n \qquad \text{for } 1 \leqslant n \leqslant N \tag{3.13}$$

The resulting slopes γ_N are shown in Fig. 5 for the logistic equation (1.2) at $b = b_{1,\text{cr}}$ and at $b = b_{3,\text{cr}}$, and for Eq. (1.3) at $b = b_{1,\text{cr}}$. In all these cases, we have performed 10^4 runs.

For the curves 1–3, the initial distribution has been chosen constant for $0 \leqslant x_0 \leqslant 1$. We see that the two limit points of period 2^k cycles behave similarly, but at $b_{3,\text{cr}}$ the behavior is completely different: $|\partial x_n/\partial x_0|$ increases at first dramatically, and only for very large n one observes the expected power behavior. This is not surprising. The attractor for Eq. (1.2) at $b = b_{3,\text{cr}}$, shown in Fig. 4, lies in three relatively narrow bands, each of which qualitatively resembles Fig. 2. If x_0 is in one of these bands, x_{3n} approaches the attractor in the way described above. If not, the first few iterates will jump essentially chaotically, until the first x_n falls into one of the bands. During this chaotic evolution, which lasts in the mean for ~ 15 iterations, $|\partial x_n/\partial x_0|$ increases roughly exponentially, with an effective Lyapounov exponent of ~ 0.5. Only for $n \gg 15$, we observe the universal power behavior.

In order to check this, we have also performed calculations with

$$w_0(x) = 10 \cdot \Theta(0.05 - |x - 0.5|) \tag{3.14}$$

In this way, we strongly suppress the transient chaotic phase, and the resulting curve (4) in Fig. 4 is indeed similar to the curves obtained for the period 2^k cycles.

The value of γ resulting from these calculations is $\gamma = 0.60 \pm 0.01$.

(b) Next, we want to study the asymptotic behavior of

$$\Delta_n \equiv |x_n - y_n| \qquad \text{for } \Delta_0 \ll 1$$

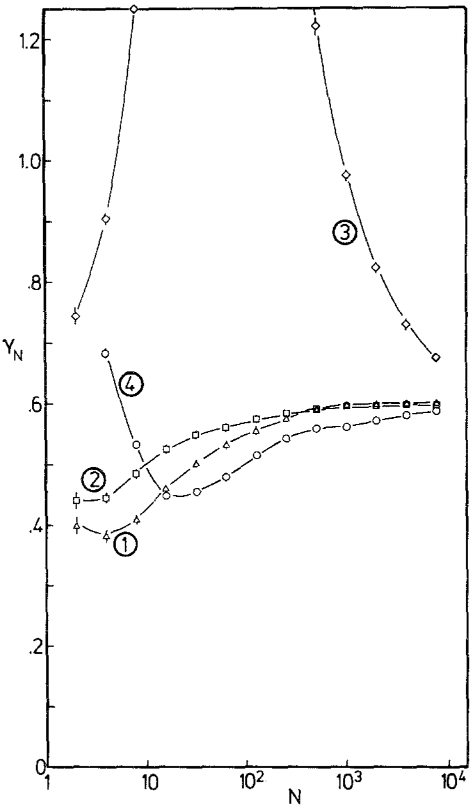

Fig. 5. Coefficients γ_N in linear fits, Eq. (3.13), to $\ln|\partial x_n/\partial x_0|$. The curves are drawn only for guidance. Curve 1: Eq. (1.2), $b = b_{1,\text{cr}} = 3.56994567\ldots$, $w_0(x) = \theta(x)\cdot\theta(1-x)$. Curve 2: Eq. (1.3), $b = b_{1,\text{cr}} = 2.3002283\ldots$, $w_0(x) = \theta(x)\cdot\theta(1-x)$. Curve 3: Eq. (1.2), $b = b_{3,\text{cr}} = 3.84943368\ldots$, $w_0(x) = \theta(x)\cdot\theta(1-x)$. Curve 4: Eq. (1.2), $b = b_{3,\text{cr}}$, $w_0(x) = 10.\ \theta(0.05 - |x-0.5|)$.

For sufficiently small Δ_0, the two points x_0 and y_0 will be in the same interval $I_{k,i}$. Similarly, x_n and y_n will be in the same $I_{k,i}$, as long as Δ_n is much smaller than the average distance d_n from the attractor discussed in the last section. In this case Δ_n will increase like $\Delta_n \sim \Delta_0 n^\gamma$. Since on the other hand $d_n \sim n^{-\epsilon}$, both lengths will become comparable for

$$n \approx \bar{n} \equiv \Delta_0^{-1/(\epsilon+\gamma)} \tag{3.15}$$

After this, Δ_n can no longer increase since x_n and y_n are bound to remain in common intervals $J_{k,i}$ with $k \approx (\ln \bar{n})/(\ln 2)$, the mean length of which is of the same order of magnitude as $d_{\bar{n}}$. Thus we finally obtain

$$\langle\Delta_n\rangle \sim \begin{cases} \Delta_0 \cdot n^{\gamma} & \text{for } n \ll \bar{n} \\ \Delta_0^{\epsilon/(\epsilon+\gamma)} & \text{for } n \gg \bar{n} \end{cases} \tag{3.16}$$

Again, we have checked this by Monte Carlo calculations. Results for Eq. (1.2) with $b = b_{1,\mathrm{cr}}$ are shown in Fig. 6. There, we have plotted the geometric average distance for fixed Δ_0, averaged over 10^4 runs and over $2^k \leqslant n < 2^{k+1}$, $k = 0, 1, 2, \ldots, 10$. We clearly see the expected behavior. For large n we get $\langle\Delta_n\rangle \sim \Delta_0^{0.77\pm0.01}$, which is in perfect agreement with Eq. (3.16) and the previous values for ϵ and γ.

(c) As a last topic, let us study the distribution of $|\partial x_n/\partial x_0|$. As we have already pointed out, its arithmetic average increases faster than a power of n. Thus, the distribution must be rather broad.

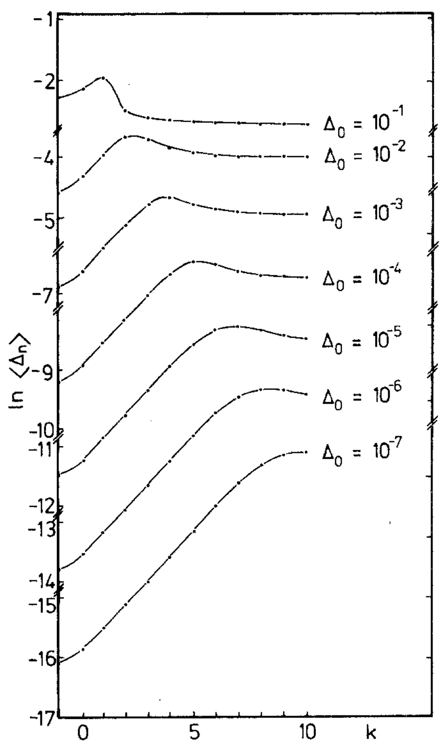

Fig. 6. Geometric average values of $\Delta_n = |x_n - y_n|$ for Eq. (1.2) at $b = b_{1,\mathrm{cr}}$. Averages are taken over $0 < x_0 < 1 - \Delta_0$, $y_0 = x_0 + \Delta_0$ [with $w_0(x_0) = \mathrm{const}$], and over $2^k \leqslant n < 2^{k+1}$. Again, the lines are drawn only to guide one's eyes. The leftmost points represent Δ_0.

In the next section we shall present a non-deterministic model which shows all essential features discussed above and which also allows to calculate the asymptotic behavior of all cumulants of $\ln|\partial x_n/\partial x_0|$,

$$\mu_1(n) = \int dx_0\, w_0(x_0) \ln\left|\frac{\partial x_n}{\partial x_0}\right|$$

$$\mu_2(n) = \int dx_0\, w_0(x_0)\left[\ln\left|\frac{\partial x_n}{\partial x_0}\right|\right]^2 - \mu_1^2(n) \tag{3.17}$$

$$\mu_3(n) = \int dx_0\, w_0(x_0)\left[\ln\left|\frac{\partial x_n}{\partial x_0}\right|\right]^3 - 3\mu_2(n)\,\mu_1(n) - \mu_1^3(n)$$

etc. We shall find that

$$\mu_k(n) \underset{n\to\infty}{\sim} \gamma_k \cdot \ln n \tag{3.18}$$

with

$$\gamma_k \underset{k\to\infty}{\sim} \text{const} \times (k-1)! \tag{3.19}$$

Thus all cumulants are proportional to $\ln n$, but the coefficients diverge so rapidly that the arithmetic mean of $|\partial x_n/\partial x_0|$, given by

$$\int dx_0\, w_0(x)\left|\frac{\partial x_n}{\partial x_0}\right| = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k!}\,\mu_k(n)\right] \tag{3.20}$$

increases faster than any power of n.

Unfortunately, we were not able to verify this by more rigorous arguments, but we have performed again Monte Carlo calculations, for Eq. (1.2) with $b = b_{1,\mathrm{cr}}$.

The Cesaro averages

$$\bar{\mu}_k(n) = \frac{1}{n}\sum_{\nu=1}^{n} \mu_k(\nu) \tag{3.21}$$

are shown in Fig. 7 for $k = 1$ to 4, and $n \leqslant 8192$. The results are based on 10^4 runs with a flat distribution of x_0 in $[0, 1]$. Indeed, we have performed the averages independently for $x_0 \in [0.25, 0.75]$ and for $x_0 \in [0, 0.25] \cup [0.75, 1]$. As expected, the differences due to these different initial conditions vanish for large n.

While the first two cumulants clearly increase linearly with $\ln n$, such a linear increase is not evident for $\bar{\mu}_3$ and $\bar{\mu}_4$. This is not surprising as nonleading effects are larger for larger k. The slopes for large n are, however, compatible with Eq. (3.19), for $k \geqslant 2$: the broken lines in Fig. 7 have slopes $2.05 \times (k-1)!$, and fit the behavior of $\bar{\mu}_k(n)$ rather well.

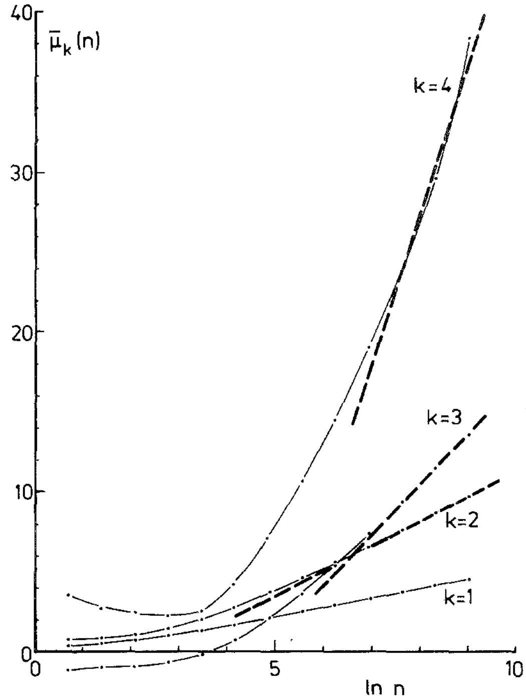

Fig. 7. Cesaro averages of the cumulants $\mu_k(n)$ of $\ln|\partial x_n/\partial x_0|$. The broken lines have slopes const $\times (k-1)!$, as predicted for the slopes of $\mu_k(n)$ when $n \to \infty$.

4. A PROBABILISTIC MODEL

For critical mappings (1.1), the evolution $x_0 \to x_1 \to \cdots \to x_n \to \cdots$ is, of course, deterministic. Furthermore, the dependence on the initial conditions is not sensitive in the sense of Ref. 11. However, we have seen that some quantities are so sensitive that it does not seem practical to follow the evolution exactly. Instead, we shall present in this section a nondeterministic (i.e., probabilistic) model which shows all essential features found above.

After n iterations of a randomly chosen x_0, x_n will be located in one of the intervals $I_{k,i}$.[2] If $i < 2^k$, it will be mapped into $I_{k,i+1}$ in the next iteration. If $i = 2^k$, it will be mapped either into $I_{k,1}$ or into some $I_{k',i'}$ with $k' > k$. Which of these two possibilities occurs, and what are k' and i' in the latter case, will depend on x_0 in a very sensitive way. Thus in our model

[2] The chance that it is exactly on the attractor is zero.

we shall assume that the jump from I_k to $I_{k'}$ occurs randomly, with given probabilities, but independently of the exact value of x_n. Consequently, the state of the system is fully described by the index k of the interval $I_{k,i} \ni x_n$, and we assume that the evolution of the probabilities $P_k(n) = \text{prob}(x_n \in I_k)$ is Markoffian,

$$P_k(n+1) = \sum_{k' \leqslant k} W_{k,k'} P_k(n) \tag{4.1}$$

It is easy to see that, asymptotically, $P_k(n)$ must decrease exponentially for fixed k,

$$P_k(n) \underset{\substack{n \to \infty \\ k \text{ fixed}}}{\sim} c^{-n/2^k} \tag{4.2}$$

with c given by Eq. (3.11); in fact, the tail of $P_k(n)$ for large n is dominated by orbits which are close to the kth periodic orbit for most iterations. This implies that

$$1 - W_{k,k} = \sum_{k'=k+1}^{\infty} W_{k',k} \underset{k \to \infty}{\approx} \frac{\ln c}{2^k} \tag{4.3}$$

For simplicity and definiteness, we shall take

$$W_{k,k'} = \frac{\ln c}{2^k} \delta_{k,k'+1} + \left(1 - \frac{\ln c}{2^k}\right) \delta_{k,k'} \tag{4.4}$$

Let us now assume that $x_0 \in I_{k_0}$, i.e., $P_k(0) = \delta_{k,k_0}$. In order to get the asymptotic behavior of $P_k(n)$, we need the eigenvalues and eigenvectors of $W_{k,k'}$. The eigenvalues are

$$\lambda_i = 1 - \frac{\ln c}{2^i}, \qquad i = 0, 1, 2, \ldots \tag{4.5}$$

The right-eigenvector u^0 corresponding to λ_0 is easily found to be

$$u_k^0 = \frac{(-2)^k}{(2^k - 1)(2^{k-1} - 1) \cdots (2-1)} \equiv r_k, \qquad k = 0, 1, 2, \ldots \tag{4.6}$$

The other right-eigenvectors u^i are simply

$$u_k^i = \begin{cases} r_{k-i} & \text{for } k \geqslant i \\ 0 & \text{for } k < i \end{cases} \tag{4.7}$$

The left-eigenvectors v^i are

$$v_k^i = \begin{cases} 0 & \text{for } k > i \\ s_{i-k} & \text{for } k \leqslant i \end{cases} \tag{4.8}$$

with

$$s_0 = 1 \tag{4.9a}$$

and

$$s_k = \left[(1 - 2^{-k})(1 - 2^{-k+1}) \cdots (1 - 2^{-1}) \right]^{-1} \tag{4.9b}$$

Notice that indeed $(v^i, u^j) = \delta_{ij}$.

Using these results, we get

$$\begin{aligned} P_k(n) &= (W^n)_{k,k_0} \\ &= \sum_{i=0}^{\infty} u_k^i \lambda_i^n v_{k_0}^i \\ &= \sum_{i=k_0}^{k} r_{k-i} \left(1 - \frac{\ln c}{2^i}\right)^n s_{i-k_0} \end{aligned} \tag{4.9c}$$

The asymptotic behavior for $n \to \infty$ and $k \gg k_0$ is dominated by contributions with $i \approx k$. Using the fact that

$$\begin{aligned} \lim_{i \to \infty} s_i &= 1 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{16}{15} \cdot \cdots \\ &= 3.463 \cdots \equiv \beta \end{aligned} \tag{4.10}$$

we finally get

$$\begin{aligned} P_{k_1}(n) &\underset{n \to \infty}{\approx} \beta \sum_{j=0}^{\infty} r_j c^{-n 2^{j-k}} \\ &\equiv p\left(k - \frac{\ln n}{\ln 2}\right) \end{aligned} \tag{4.11}$$

This shows all wanted properties. First, it is a function of $n/2^k$ only, and thus it satisfies the crucial relation (2.25). Notice that the present model is a kind of one-sided random walk model (with $\ln n \leftrightarrow t$, $k \leftrightarrow x$), and it might seem strange that the evolution is dispersion-free. It follows, however, from the fact that the transition probability *per unit* $\ln n$ increases linearly with n, while it decreases exponentially with k. Secondly, it is independent of k_0: this is a necessary condition if we want universal behavior, independent of the initial distribution $w_0(x)$.

The above result implies that the model indeed shows the properties discussed in Section 2. In order to estimate the asymptotic behavior of $\partial x_n / \partial x_0$, we have to add a further assumption. Referring to the discussion leading to Eq. (3.12), we assume that

$$\left| \frac{\partial x_{n+1}}{\partial x_n} \right| \approx c^{2^{-k}} \qquad \text{for } x \in I_k \tag{4.12}$$

i.e., we replace $|\partial x_{n+1} / \partial x_n|$ by its average value at the periodic orbit in I_k.

The cumulants $\mu_k(n)$ are obtained from the generating function

$$G(z,n) = \exp\left[\sum_{k=1}^{\infty} \frac{z^k}{k!} \mu_k(n)\right] = \int dx_0\, w_0(x_0) \left|\frac{\partial x_n}{\partial x_0}\right|^z \tag{4.13}$$

In our Markoffian model we get

$$\begin{aligned} G(z,n) &= \sum_{k_n \geqslant k_{n-1} \geqslant \cdots \geqslant k_0} W_{k_n,k_{n-1}} \cdot W_{k_{n-1},k_{n-2}} \cdots W_{k_1,k_0} \\ &\quad \times P_{k_0}(0) \cdot c^{z(2^{-k_0}+2^{-k_1}+\cdots+2^{-k_{n-1}})} \\ &= \sum_{k \geqslant k_0} \left[\tilde{W}(z)\right]_{k,k_0} P_{k_0}(0) \end{aligned} \tag{4.14}$$

with

$$\tilde{W}_{k,k'}(z) = W_{k,k'} \cdot c^{z2^{-k'}} \tag{4.15}$$

The eigenvalues of $\tilde{W}$ are simply

$$\tilde{\lambda}_i(z) = \left(1 - \frac{\ln c}{2^i}\right) c^{z/2^i} \tag{4.16}$$

The eigenvectors are less simple, but for an estimate of the asymptotic behavior we can restrict ourselves to large k_0 and large i. There,

$$\tilde{u}_k^i(z) \approx r_{k-1} \cdot (1-z)^{-k} \tag{4.17}$$

and

$$\tilde{v}_k^i(z) \approx s_{i-k} \cdot (1-z)^k \tag{4.18}$$

The generating function then becomes

$$G(z,n) \underset{n\to\infty}{\approx} \sum_{k=k_0}^{\infty} (1-z)^{k_0-k} P_k((1-z)n) \tag{4.19}$$

Since the width (in k) of the distribution $P_k(n)$ stays constant for $n \to \infty$, we can approximate it by a δ function, and finally obtain

$$G(z,n) \underset{n\to\infty}{\approx} (1-z)^{\ln n/\ln 2} \times \text{(some function of } z\text{)} \tag{4.20}$$

from which we get

$$\mu_k(n) \approx \frac{k!}{k \ln 2} \cdot \ln n \tag{4.21}$$

The factor $1/\ln 2 \approx 1.44$ is, of course, an artifact of the simple Ansätze (4.4) and (4.12), but otherwise both the linear behavior in $\ln n$ and the increase $\sim(k-1)!$ seem to be realistic.

5. CONCLUSIONS

We have shown that the long-time behavior of the evolution under critical mappings is governed by scaling laws reminiscent of critical behavior ("critical slowing down"). Our arguments are only heuristic, but they are supported by extensive Monte Carlo calculations. Our results underline once more the similarity between critical mappings and other critical phenomena.

Critical maps are not sensitively dependent on initial conditions in the sense of Ref. 11. But as we approach the attractor closer and closer, finer and finer details are relevant for the further approach which thus becomes sensitively dependent on x_0.

In experimental situations, this would mean that the approach towards the attractor, and the increase of the distance between two close-by points, are essentially random processes. Indeed, we have found that all our main results follow already from a simple probabilistic model which depicts the evolution as a Markoffian random walk.

APPENDIX

In this Appendix we shall study the asymptotic behavior of the total variation Δ_n of the nth iterate of the Feigenbaum[6] scaling function $g(x)$ between 0 and 1,

$$\Delta_n \equiv \int_0^1 dx \left| \frac{d}{dx} g^{(n)}(x) \right| \tag{A.1}$$

We denote by x_0 ($= 0.832 \ldots$) the zero of $g(x)$ in this interval, and write

$$\Delta_n = \Delta_n^{(1)} + \Delta_n^{(2)} \tag{A.2}$$

with

$$\Delta_n^{(1)} = \int_0^{x_0} dx \left| \frac{d}{dx} g^{(n)}(x) \right| \tag{A.3}$$

$$\Delta_n^{(2)} = \int_{x_0}^1 dx \left| \frac{d}{dx} g^{(n)}(x) \right|$$

Notice that the n extrema of $g^{(n)}(x)$ on $[0, 1[$ are precisely the n first iterates of the point $x = 0$ which satisfy the Feigenbaum scaling relation

$$g^{(2n)}(0) = -\frac{1}{\alpha} g^{(n)}(0), \qquad \alpha = 2.5029 \cdots \tag{A.4}$$

From this one easily sees that for $n \geqslant 1$

$$\Delta_{n+1}^{(1)} = \Delta_n \tag{A.5}$$

and

$$\Delta_{2n+1}^{(2)} = \frac{1}{\alpha} \Delta_n \tag{A.6}$$

Furthermore,

$$0 < \Delta_{2n}^{(2)} < \Delta_{2n+1}^{(2)} \tag{A.7}$$

so that

$$\Delta_{2n+1} = \Delta_{2n} + \frac{1}{\alpha} \Delta_n \tag{A.8}$$

and

$$\Delta_{2n-1} < \Delta_{2n} < \Delta_{2n-1} + \frac{1}{\alpha} \Delta_n \tag{A.9}$$

Let us define δ_n $(n = 1, 2, \ldots)$ by

$$\delta_1 = 1 + \frac{1}{\alpha} \tag{A.10}$$

and

$$\delta_n = \delta_{n-1} + \frac{1}{\alpha} \delta_{[n/2]}, \qquad n \geqslant 2 \tag{A.11}$$

where $[a]$ is the largest integer $\leqslant a$. Then

$$\delta_{[(n+1)/2]} \leqslant \Delta_n \leqslant \delta_n \tag{A.12}$$

From Eq. (A.11) one verifies that, for large n, roughly

$$\delta_n \sim \exp\left[\frac{(\ln n)^2}{2 \ln 2} \right] \tag{A.13}$$

and hence Eq. (1.13) holds at least for $F(x, b_{\mathrm{cr}}) = g(x)$.

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